Solutions

4.3: Matrix Inversion

Definition 1. The <u>inverse</u> of an $n \times n$ matrix A is the $n \times n$ matrix A^{-1} such that

$$AA^{-1} = A^{-1}A = I.$$

If the inverse of A exists, it is said to be <u>invertible.</u> Otherwise, it is said to be <u>singular</u>.

Example 1. Is $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ invertible?

$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ 0 & 0 \end{bmatrix}$. No way this can be the identity.

Example 2. Compute the inverse of the following matrices.

(a) $P = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$

(b) $Q = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -2 & -1 \\ 3 & 0 & 0 \end{bmatrix}$

(a) $\left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|cc} 1 & 0 & 1/2 & -1/2 \\ 0 & 1 & -1/2 & -1/2 \end{array}\right]$

So $P^{-1} = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & -1/2 \end{bmatrix}$

(b) $\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & -2 & -1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & -1/2 & -1/2 & 1/2 \\ 0 & 0 & 1 & 1 & 0 & -1/3 \end{array}\right]$

So $Q^{-1} = \begin{bmatrix} 0 & 0 & 1/3 \\ -1/2 & -1/2 & 1/2 \\ 1 & 0 & -1/3 \end{bmatrix}$.

Theorem 1. The inverse of a 2×2 matrix is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

provided $ad - bc \neq 0$. The quantity $ad - bc$ is called the <u>determinant</u> of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If the determinant is 0, then the matrix is singular.

Example 3. Determine if the matrix $S = \begin{bmatrix} 1 & 1 & 2 \\ -2 & 0 & 4 \\ 3 & 1 & -2 \end{bmatrix}$ is invertible by looking at the reduced row echelon form.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ -2 & 0 & 4 & 0 & 1 & 0 \\ 3 & 1 & -2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 0 & -1/2 & 0 \\ 0 & 1 & 4 & 0 & 3/2 & 1 \\ 0 & 0 & 0 & 1 & -1 & -1 \end{array}\right]$$

S is not invertible.

Example 4. Solve the system of equations

$$2x + z = 1, \quad 2x + y - z = 1, \quad 3x + y - z = 1$$

using matrix inversion.

$A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & -1 \\ 3 & 1 & -1 \end{bmatrix}$. We want to find $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \bar{x}$

such that

$A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Let $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \bar{y}$. We want $A\bar{x} = \bar{y}$.

$\bar{x} = A^{-1}\bar{y} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$ so $(x, y, z) = (0, 2, 1)$.